This paper develops tests for roots in linear time series which have a modulus of one but which correspond to seasonal frequencies. Critical values for the tests are generated by Monte Carlo methods or are shown to be available from Dickey–Fuller or Dickey–Hasza–Fuller critical values. Representations for multivariate processes with combinations of seasonal and zero-frequency unit roots are developed leading to a variety of autoregressive and error-correction representations. The techniques are used to examine cointegration at different frequencies between consumption and income in the U.K.

1. Introduction

The rapidly developing time-series analysis of models with unit roots has had a major impact on econometric practice and on our understanding of the response of economic systems to shocks. Univariate tests for unit roots were first proposed by Fuller (1976) and Dickey and Fuller (1979) and were applied to a range of macroeconomic data by Nelson and Plosser (1982). Granger (1981) proposed the concept of cointegration which recognized that even though several series all had unit roots, a linear combination could exist which would not. Engle and Granger (1987) present a theorem giving several representations of cointegrated series and tests and estimation procedures. The testing is a direct generalization of Dickey and Fuller to the hypothesized linear combination.

All of this work assumes that the root of interest not only has a modulus of one, but is precisely one. Such a root corresponds to a zero-frequency peak in

*The research was carried out while the first author was on sabbatical at UCSD and the last author was completing his dissertation. The authors are indebted to the University of Aarhus, NSF SES87-05884, and SES87-04669 for financial support. The data will be made available through the Inter-university Consortium for Political and Social Research at the University of Michigan.
the spectrum. Furthermore, it assumes that there are no other unit roots in the system. Because many economic time series exhibit substantial seasonality, there is a definite possibility that there may be unit roots at other frequencies such as the seasonals. In fact, Box and Jenkins (1970) and the many time-series analysts influenced by their work implicitly assume that there are seasonal unit roots by using the seasonal differencing filter.

This paper describes in section 2 various classes of seasonal processes and in section 3 sets out to test for seasonal unit roots in time-series data both in the presence of other unit roots and other seasonal processes. Section 4 defines seasonal cointegration and derives several representations. Section 5 gives an empirical example and section 6 concludes.

2. Seasonal time-series processes

Many economic time series contain important seasonal components and there are a variety of possible models for seasonality which may differ across series. A seasonal series can be described as one with a spectrum having distinct peaks at the seasonal frequencies \( \omega_s = 2\pi j/s \), \( j = 1, \ldots, s/2 \), where \( s \) is the number of time periods in a year, assuming \( s \) to be an even number and that a spectrum exists. In this paper, quarterly data will be emphasised so that \( s = 4 \), but the results can be naturally extended in a straightforward fashion to monthly data, for example.

Three classes of time-series models are commonly used to model seasonality. These can be called:

(a) Purely deterministic seasonal processes,
(b) Stationary seasonal processes,
(c) Integrated seasonal processes,

Each is frequently used in empirical work often with an implicit assumption that they are all equivalent. The first goal of this paper is to develop a testing procedure which will determine what class of seasonal processes is responsible for the seasonality in a univariate process. Subsequently this approach will deliver multivariate results on cointegration at seasonal frequencies.

A purely deterministic seasonal process is a process generated by seasonal dummy variables such as the following quarterly series:

\[
x_t = \mu_t \quad \text{where} \quad \mu_t = m_0 + m_1 S_{1t} + m_2 S_{2t} + m_3 S_{3t}.
\] (2.1)

Notice that this process can be perfectly forecast and will never change its shape.
A stationary seasonal process can be generated by a potentially infinite autoregression

\[ \varphi(B)x_t = \epsilon_t, \quad \epsilon_t \text{ i.i.d.,} \]

with all of the roots of \( \varphi(B) = 0 \) lying outside the unit circle but where some are complex pairs with seasonal periodicities. More precisely, the spectrum of such a process is given by

\[ f(\omega) = \sigma^2 / |\varphi(e^{i\omega})|^2, \]

which is assumed to have peaks at some of the seasonal frequencies \( \omega_j \). An example for quarterly data is

\[ x_t = \rho x_{t-4} + \epsilon_t, \]

which has a peak at both the seasonal periodicities \( \pi/2 \) (one cycle per year) and \( \pi \) (two cycles per year) as well as at zero frequency (zero cycles per year).

A series \( x_t \) is an integrated seasonal process if it has a seasonal unit root in its autoregressive representation. More generally it is integrated of order \( d \) at frequency \( \theta \) if the spectrum of \( x_t \) takes the form

\[ f(\omega) \sim c(\omega - \theta)^{-2d}, \]

for \( \omega \) near \( \theta \). This is conveniently denoted by

\[ x_t \sim I_\theta(d). \]

The paper will concentrate on the case \( d = 1 \). An example of an integrated quarterly process at two cycles per year is

\[ x_t = -x_{t-1} + \epsilon_t, \quad (2.2) \]

and at one cycle per year it is

\[ x_t = -x_{t-2} + \epsilon_t. \quad (2.3) \]

The very familiar seasonal differencing operator, advocated by Box and Jenkins (1970) and used as a seasonal process by Grether and Nerlove (1970) and Bell and Hillmer (1985) for example, can be written as

\[
(1 - B^4)x_t = \epsilon_t = (1 - B)(1 + B + B^2 + B^3)x_t,
= (1 - B)(1 + B)(1 + B^2)x_t,
= (1 - B)S(B)x_t, \quad (2.4)
\]
which therefore has four roots with modulus one: one is a zero frequency, one at two cycles per year, and two complex pairs at one cycle per year.

The properties of seasonally integrated series are not immediately obvious but are quite similar to the properties of ordinary integrated processes as established for example by Fuller (1976). In particular they have 'long memory' so that shocks last forever and may in fact change permanently the seasonal patterns. They have variances which increase linearly since the start of the series and are asymptotically uncorrelated with processes with other frequency unit roots.

The generating mechanisms being considered, such as (2.2) or (2.4), are stochastic difference equations. They generalize the ordinary $I(1)$, or $I_0(1)$ in the present notation, process. It is well known that an equation of the form

$$(1 - B)x_t = \varepsilon_t,$$  \hspace{1cm} (2.5)

has two components to its solution: the homogeneous solution $x_{1t}$ where

$$(1 - B)x_{1t} = 0$$

and the particular solution $x_{2t}$, given by

$$x_{2t} = \left(\frac{1}{1 - B}\right)\varepsilon_t.$$  

Thus $x_t = x_{1t} + x_{2t}$, where $x_{1t} = x_0$ (the starting value) and $x_{2t} = \sum_{j=0}^{t-1} \varepsilon_{t-j}$.

Clearly, if $E[\varepsilon_t] = m \neq 0$, then $x_{2t}$ will contain a linear trend $mt$.

The equation with $S(B) = (1 + B)(1 + B^2)$,

$$S(B)x_t = \varepsilon_t,$$  \hspace{1cm} (2.6)

also has a solution with two components. The homogeneous solution is

$$x_{1t} = c_1(-1)^t + c_2(i)^t + c_3(-i)^t,$$

where $c_1, c_2, c_3$ are determined from the starting conditions, plus the requirement that $x_{1t}$ is a real series, i.e., $c_2$ and $c_3$ are complex conjugates. If $x_{-2} = x_{-1} = x_0 = 0$ so that the starting values contain no seasonal, then $x_{1t} \equiv 0$.

The particular solution is

$$x_{2t} = \left[S(B)\right]^{-1}\varepsilon_t,$$
and noting that

\[ \left[ S(B) \right]^{-1} = \frac{1}{2} \left[ \frac{1}{1 + B} + \frac{1 - B}{1 + B^2} \right], \]

some algebra gives

\[
x_{2t} = \frac{1}{2} \sum_{j=0}^{t-1} (-1)^j \varepsilon_{t-j} + \sum_{j=0}^{\text{int}((t-1)/2)} (-1)^j \Delta \varepsilon_{t-2j},
\]

where \( \Delta = 1 - B \) and \( \text{int}[z] \) is the largest integer in \( z \). The two parts of this solution correspond to the two seasonal roots and to eqs. (2.2) and (2.3).

The homogeneous solutions to eqs. (2.9), (2.2), and (2.3) are given, respectively, by

\[
s_{1t} = \sum_{j=0}^{t-1} \varepsilon_{t-j} \quad \text{for zero-frequency root},
\]

\[
s_{2t} = \sum_{j=0}^{t-1} (-1)^j \varepsilon_{t-j} \quad \text{for the two-cycle-per-year root},
\]

\[
s_{3t} = \sum_{j=0}^{\text{int}((t-1)/2)} (-1)^j \Delta \varepsilon_{t-2j} \quad \text{for the one-cycle-per-year root}.
\]

The variances of these series are given by

\[ V(s_{1t}) = V(s_{2t}) = V(s_{3t}) = t \sigma^2, \]

so that all of the unit roots have the property that the variance tends to infinity as the process evolves. When the series are excited by the same \( \{ \varepsilon_t \} \) and \( t \) is divisable by four, the covariances are all zero. At other values of \( t \) the covariances are at most \( \sigma^2 \), so the series are asymptotically uncorrelated as well as being uncorrelated in finite samples for complete years of data.

It should be noted that, if \( \mathbb{E}[\varepsilon_t] = m \neq 0 \), all \( t \), then the first term in \( x_{2t} \) will involve an oscillation of period 2. The complete solution to (2.6) contains both cyclical deterministic terms, corresponding to ‘seasonal dummies’ plus long nondeclining sums of past innovations or their changes. Thus, a series generated by (2.6) will have a component that is seasonally integrated and may also have a deterministic seasonal component, largely depending on the starting values. A series generated by (2.6) will be inclined to have a seasonal with peak that varies slowly through time, but if the initial deterministic component is large, it may not appear to drift very fast.
If \( x_t \) is generated by
\[
(1 - B^4) x_t = \epsilon_t, \tag{2.7}
\]
the equation will have solutions that are linear combinations of those for (2.5) and (2.6).

A series with a clear seasonal may be seasonally integrated, have a deterministic seasonal, a stationary seasonal, or some combination. A general class of linear time-series models which exhibit potentially complex forms of seasonality can be written as
\[
d(B) a(B)(x_t - \mu_t) = \epsilon_t, \tag{2.8}
\]
where all the roots of \( a(z) = 0 \) lie outside the unit circle, all the roots of \( d(z) = 0 \) lie on the unit circle, and \( \mu_t \) is given as above. Stationary seasonality and other stationary components of \( x \) are absorbed into \( a(B) \), while deterministic seasonality is in \( \mu_t \) when there are no seasonal unit roots in \( d(B) \). Section 3 of this paper considers how to test for seasonal unit roots and zero-frequency unit roots when other unit roots are possibly present and when deterministic or stochastic seasonals may be present.

A pair of series each of which are integrated at frequency \( \omega \) are said to be cointegrated at that frequency if a linear combination of the series is not integrated at \( \omega \). If the linear combination is labeled \( \alpha \), then we use the notation
\[
x_t \sim CI_{\omega}, \quad \text{with cointegrating vector } \alpha.
\]
This will occur if, for example, each of the series contains the same factor which is \( I_{\omega}(1) \). In particular, if
\[
x_t = \alpha v_t + \bar{x}_t, \quad y_t = v_t + \bar{y}_t,
\]
where \( v_t \) is \( I_{\omega}(1) \) and \( \bar{x}_t \) and \( \bar{y}_t \) are not, then \( z_t = x_t - \alpha y_t \) is not \( I_{\omega}(1) \), although it could be still integrated at other frequencies. If a group of series are cointegrated, there are implications about their joint generating mechanism. These are considered in section 4 of this paper.

### 3. Testing for seasonal unit roots

It is the goal of the testing procedure proposed in this paper to determine whether or not there are any seasonal unit roots in a univariate series. The test must take seriously the possibility that seasonality of other forms may be present. At the same time, the tests for conventional unit roots will be examined in seasonal settings.
In the literature there exist a few attempts to develop such tests. Dickey, Hasza, and Fuller (1984), following the lead suggested by Dickey and Fuller for the zero-frequency unit-root case, propose a test of the hypothesis \( a = 1 \) against the alternative \( a < 1 \) in the model \( x_t = ax_{t-s} + \epsilon_t \). The asymptotic distribution of the least-squares estimator is found and the small-sample distribution obtained for several values of \( s \) by Monte Carlo methods. In addition the test is extended to the case of higher-order stationary dynamics. A major drawback of this test is that it doesn’t allow for unit roots at some but not all of the seasonal frequencies and that the alternative has a very particular form, namely that all the roots have the same modulus. Exactly the same problems are encountered by the tests proposed by Bhargava (1987). In Ahtola and Tiao (1987) tests are proposed for the case of complex roots in the quarterly case but also their suggestion may at best be a part of a more comprehensive test strategy. In this paper we propose a test and a general framework for a test strategy that looks at unit roots at all the seasonal frequencies as well as the zero frequency. The test follows the Dickey–Fuller framework and in fact has a well-known distribution possibly on transformed variables in some special cases.

For quarterly data, the polynomial \((1 - B^4)\) can be expressed as

\[
(1 - B^4) = (1 - B)(1 + B)(1 - iB)(1 + iB)
= (1 - B)(1 + B)(1 + B^2),
\] (3.1)

so that the unit roots are 1, \(-1\), \(i\), and \(-i\) which correspond to zero frequency, \(\frac{1}{2}\) cycle per quarter or 2 cycles per year, and \(\frac{1}{4}\) cycle per quarter or one cycle per year. The last root, \(i\), is indistinguishable from the one at \(i\) with quarterly data (the aliasing phenomenon) and is therefore also interpreted as the annual cycle.

To test the hypothesis that the roots of \(q(B)\) lie on the unit circle against the alternative that they lie outside the unit circle, it is convenient to rewrite the autoregressive polynomial according to the following proposition which is originally due to Lagrange and is used in approximation theory.

**Proposition.** Any (possibly infinite or rational) polynomial \(q(B)\), which is finite-valued at the distinct, nonzero, possibly complex points \(\theta_1, \ldots, \theta_p\), can be expressed in terms of elementary polynomials and a remainder as follows:

\[
q(B) = \sum_{k=1}^{p} \lambda_k \Delta(B)/\delta_k(B) + \Delta(B)q**(B),
\] (3.2)

where the \(\lambda_k\) are a set of constants, \(q**(B)\) is a (possibly infinite or rational)
polynomial, and
\[ \delta_k(B) = 1 - \frac{1}{\theta_k} B, \quad \Delta(B) = \prod_{k=1}^{p} \delta_k(B). \]

Proof. Let \( \lambda_k \) be defined to be
\[ \lambda_k = \varphi(\theta_k)/\prod_{j \neq k} \delta_j(\theta_k), \]
which always exists since all the roots of the \( \delta \)'s are distinct and the polynomial is bounded at each value by assumption. The polynomial
\[ \varphi(B) - \sum_{k=1}^{p} \lambda_k \Delta(B)/\delta_k(B) = \varphi(B) - \sum_{k=1}^{p} \varphi(\theta_k) \prod_{j \neq k} \delta_j(B)/\delta_j(\theta_k) \]
will have zeroes at each point \( B = \theta_k \). Thus it can be written as the product of a polynomial, say \( \varphi^*(B) \), and \( \Delta(B) \). QED

An alternative and very useful form of this expression is obtained by adding and subtracting \( \Delta(B) \sum \lambda_k \) to (3.2) to get
\[ \varphi(B) = \sum_{k=1}^{p} \lambda_k \Delta(B)(1 + \delta_k(B))/\delta_k(B) + \Delta(B) \varphi^*(B), \quad (3.3) \]
where \( \varphi^*(B) = \varphi^{**}(B) + \sum \lambda_k \). In this representation \( \varphi(0) = \varphi^*(0) \) which is normalized to unity.

It is clear that the polynomial \( \varphi(B) \) will have a root at \( \theta_k \) if and only if \( \lambda_k = 0 \). Thus testing for unit roots can be carried out equivalently by testing for parameters \( \lambda = 0 \) is an appropriate expansion.

To apply this proposition to testing for seasonal unit roots in quarterly data, expand a polynomial \( \varphi(B) \) about the roots +1, -1, \( i \), and \( -i \) as \( \theta_k \), \( k = 1, \ldots, 4 \). Then, from (3.3),
\[ \varphi(B) = \lambda_1 B(1 + B)(1 + B^2) + \lambda_2 (-B)(1 - B)(1 + B^2) \]
\[ + \lambda_3 (-iB)(1 - B)(1 + B)(1 - iB) \]
\[ + \lambda_4 (iB)(1 - B)(1 + B)(1 + iB) \]
\[ + \varphi^*(B)(1 - B^4). \]
Clearly, $\lambda_3$ and $\lambda_4$ must be complex conjugates since $\varphi(B)$ is real. Simplifying and substituting $\tau_1 = -\lambda_1$, $\tau_2 = -\lambda_2$, $2\lambda_3 = -\tau_3 + i\tau_4$, and $2\lambda_4 = -\tau_3 - i\tau_4$, gives

$$
\varphi(B) = -\tau_1 B (1 + B + B^2 + B^3) - \tau_2 (-B) (1 - B + B^2 - B^3) - (\tau_3 + \tau_4 B) (-B) (1 - B^2) + \varphi^*(B) (1 - B^4). \tag{3.4}
$$

The testing strategy is now apparent. The data are assumed to be generated by a general autoregression

$$
\varphi(B) x_t = \varepsilon_t, \tag{3.5}
$$

and (3.4) is used to replace $\varphi(B)$, giving

$$
\varphi^*(B) y_{4t} = \tau_1 y_{1t-1} + \tau_2 y_{2t-1} + \tau_3 y_{3t-2} + \tau_4 y_{3t-1} + \varepsilon_t, \tag{3.6}
$$

where

$$
y_{1t} = (1 + B + B^2 + B^3) x_t = S(B) x_t, \\
y_{2t} = -(1 - B + B^2 - B^3) x_t, \\
y_{3t} = -(1 - B^2) x_t, \tag{3.7} \\
y_{4t} = (1 - B^4) x_t = \Delta_4 x_t.
$$

Eq. (3.6) can be estimated by ordinary least squares, possibly with additional lags of $y_4$ to whiten the errors. To test the hypothesis that $\varphi(\theta_k) = 0$, where $\theta_k$ is either 1, 1, or $\pm i$, one needs simply to test that $\lambda_k$ is zero. For the root 1 this is simply a test for $\tau_1 = 0$, and for $-1$ it is $\tau_2 = 0$. For the complex roots $\lambda_3$ will have absolute value of zero only if both $\tau_3$ and $\tau_4$ equal zero which suggests a joint test. There will be no seasonal unit roots if $\tau_2$ and either $\tau_3$ or $\tau_4$ are different from zero, which therefore requires the rejection of both a test for $\tau_2$, and a joint test for $\tau_3$ and $\tau_4$. To find that a series has no unit roots at all and is therefore stationary, we must establish that each of the $\tau$’s is different from zero (save possibly either $\tau_3$ or $\tau_4$). A joint test will not deliver the required evidence.

The natural alternative for these tests is stationarity. For example, the alternative to $\varphi(1) = 0$ should be $\varphi(1) > 0$ which means $\tau_1 < 0$. Similarly, the stationary alternative to $\varphi(-1) = 0$ is $\varphi(-1) > 0$ which corresponds to $\tau_2 < 0$. 
Finally, the alternative to $|\varphi(i)| = 0$ is $|\varphi(i)| > 0$. Since the null is two-dimensional, it is simplest to compute an $F$-type of statistic for the joint null, $\pi_3 = \pi_4 = 0$, against the alternative that they are not both equal to zero. An alternative strategy is to compute a two-sided test of $\pi_4 = 0$, and if this is accepted, continue with a one-sided test of $\pi_3 = 0$ against the alternative $\pi_3 < 0$. If we restrict our attention to alternatives where it is assumed that $\pi_4 = 0$, a one-sided test for $\pi_3$ would be appropriate with rejection for $\pi_3 < 0$. Potentially this could lack power if the first-step assumption is not warranted.

In the more complex setting where the alternative includes the possibility of deterministic components it is necessary to allow $\mu_t \neq 0$. The testable model becomes

$$\varphi^* (B) y_{4t} = \pi_1 y_{1t-1} + \pi_2 y_{2t-1} + \pi_3 y_{3t-2} + \pi_4 y_{3t-1} + \mu_t + \epsilon_t. \quad (3.8)$$

which can again be estimated by OLS and the statistics on the $\pi$'s used for inference.

The asymptotic distribution of the $t$-statistics from this regression were analyzed by Chan and Wei (1988). The basic finding is that the asymptotic distribution theory for these tests can be extracted from that of Dickey and Fuller (1979) and Fuller (1976) for $\pi_1$ and $\pi_2$, and from Dickey, Hasza, and Fuller (1984) for $\pi_3$ if $\pi_4$ is assumed to be zero. The tests are asymptotically similar or invariant with respect to nuisance parameters. Furthermore, the finite-sample results are well approximated by the asymptotic theory and the tests have reasonable power against each of the specific alternatives.

It is clear that several null hypotheses will be tested for each case of interest. These can all be computed from the same least-squares regression (3.6) or (3.8) unless the sequential testing of $\pi_3$ and $\pi_4$ is desired.

To show intuitively how these limiting distributions relate to the standard unit-root tests consider (3.6) with $\varphi^* (B) = 1$. The test for $\pi_1 = 0$ will have the familiar Dickey–Fuller distribution if $\pi_2 = \pi_3 = \pi_4 = 0$ since the model can be written in the form

$$y_{1t} = (1 + \pi_1) y_{1t-1} + \epsilon_t.$$ 

Similarly,

$$y_{2t} = -(1 + \pi_2) y_{2t-1} + \epsilon_t,$$

if the other $\pi$'s are zero. This is a test for a root of $-1$ which was shown by Dickey and Fuller to be the mirror of the Dickey–Fuller distribution. If $y_{2t}$ is regressed on $-y_{2t-1}$, the ordinary DF distribution will be appropriate. The
third test can be written as

\[ y_{3t} = -(1 + \pi_3) y_{3t-2} + \epsilon_t, \]

assuming \( \pi_4 = 0 \) which is therefore the mirror of the Dickey–Hasza–Fuller distribution for biannual seasonality. The inclusion of \( y_{3t-1} \) in the regression recognizes potential phase shifts in the annual component. Since the null is that \( \pi_3 = \pi_4 = 0 \), the assumption that \( \pi_4 = 0 \) may merely reduce the power of the test against some alternatives.

To show that the same distributions are obtained when it is not known \textit{a priori} that some of the \( \pi \)'s are zero, two cases must be considered. First, if the \( \pi \)'s other than the one being tested are truly nonzero, then the process does not have unit roots at these frequencies and the corresponding \( y \)'s are stationary. The regression is therefore equivalent to a standard augmented unit-root test.

If however some of the other \( \pi \)'s are zero, there are other unit roots in the regression. However, it is exactly under this condition that it is shown in section 2 that the corresponding \( y \)'s are asymptotically uncorrelated. The distribution of the test statistic will not be affected by the inclusion of a variable with a zero coefficient which is orthogonal to the included variables. For example, when testing \( \pi_1 = 0 \), suppose \( \pi_2 = 0 \) but \( y_2 \) is still included in the regression. Then \( y_1 \) and \( y_2 \) will be asymptotically uncorrelated since they have unit roots at different frequencies and both will be asymptotically uncorrelated with lags of \( y_d \) which is stationary. The test for \( \pi_1 = 0 \) will have the same limiting distribution regardless of whether \( y_2 \) is included in the regression. Similar arguments follow for the other cases.

When deterministic components are present in the regression even if not in the data, the distributions change. Again, the changes can be anticipated from this general approach. The intercept and trend portions of the deterministic mean influence only the distribution of \( \pi_1 \) because they have all their spectral mass at zero frequency. Once the intercept is included, the remaining three seasonal dummies do not affect the limiting distribution of \( \pi_1 \). The seasonal dummies, however, do affect the distribution of \( \pi_2, \pi_3, \) and \( \pi_4 \).

Table 1a gives the Monte Carlo critical values for the one-sided ‘\( t \)’ tests on \( \pi_1, \pi_2, \) and \( \pi_3 \) in the most important cases. These are very close to the Monte Carlo values from Dickey–Fuller and Dickey–Hasza–Fuller for the situations in which they tabulated the statistics.

In table 1b we present the critical values of the two-sided ‘\( t \)’ test on \( \pi_4 = 0 \) and the critical values for the ‘\( F \)’ test on \( \pi_2 \cap \pi_4 = 0 \). Notice that the distribution of the ‘\( t \)’ statistic is very similar to a standard normal except when the auxiliary regression contains seasonal dummies, in which case it becomes fatter-tailed. The distribution for the ‘\( F \)’ statistic also looks like an \( F \)
Table 1a
Critical values from the small-sample distributions of test statistics for seasonal unit roots on 24000 Monte Carlo replications: data-generating process $\Delta_4x_t = \epsilon_t \sim \text{nid}(0,1)$.

<table>
<thead>
<tr>
<th>Auxiliary regressions</th>
<th>Fractiles $t^*: \pi_1$</th>
<th>Fractiles $t^*: \pi_2$</th>
<th>Fractiles $t^*: \pi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.01</td>
<td>0.025</td>
<td>0.05</td>
</tr>
<tr>
<td>No intercept</td>
<td>-2.72</td>
<td>-2.29</td>
<td>-1.95</td>
</tr>
<tr>
<td>No seas. dum.</td>
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<td>-2.26</td>
<td>-1.97</td>
</tr>
<tr>
<td>No trend</td>
<td>-2.62</td>
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<td>-1.93</td>
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<tr>
<td></td>
<td>-2.62</td>
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<td>-1.94</td>
</tr>
<tr>
<td>Intercept</td>
<td>-3.66</td>
<td>-3.25</td>
<td>-2.96</td>
</tr>
<tr>
<td>No seas. dum.</td>
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<td>-3.14</td>
<td>-2.88</td>
</tr>
<tr>
<td>No trend</td>
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<tr>
<td></td>
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<td>-2.91</td>
</tr>
<tr>
<td>Intercept</td>
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<td>-3.56</td>
</tr>
<tr>
<td>No seas. dum.</td>
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<tr>
<td></td>
<td>-4.05</td>
<td>-3.74</td>
<td>-3.49</td>
</tr>
</tbody>
</table>
Table 1b
Critical values from the small-sample distributions of test statistics for seasonal unit roots on 24000 Monte Carlo replications: data-generating process $\Delta_4 x_t = \varepsilon_t \sim \text{nid}(0, 1)$.

<table>
<thead>
<tr>
<th>Auxiliary regressions</th>
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distribution with degrees of freedom equal to two and \( T \) minus the number of regressors in (3.6). However, when seasonal dummies are present, the tail becomes fatter here as well.

4. Error-correction representation

In this section, an error-correction representation is derived which explicitly takes the cointegrating restrictions at the zero and at the seasonal frequencies into account. As the time series being considered has poles at different locations on the unit circle, various cointegrating situations are possible. This naturally makes the general treatment mathematically complex and notationally involved. Although we treat the general case we will present the special cases considered to be of most interest.

Let \( x_t \) be a \( N \times 1 \) vector of quarterly time series, each of which potentially has unit roots at zero and all seasonal frequencies, so that each component of \((1 - B^4)x_t\) is a stationary process but may have a zero on the unit circle. The Wold representation will thus be

\[
(1 - B^4)x_t = C(B)e_t, \tag{4.1}
\]

where \( e_t \) is a vector white noise process with zero mean and covariance matrix \( \Omega \), a positive definite matrix.

There are a variety of possible types of cointegration for such a set of series. To initially examine these, apply the decomposition of (3.2) to each element of \( C(B) \). This gives

\[
C(B) = \sum_{\text{k}=1}^{p} A_k \Delta(B) / \delta_k(B) + C^{**}(B) \Delta(B),
\]

where \( \delta_k(B) = 1 - (1/\theta_k)B \) and \( \Delta(B) \) is the product of all the \( \delta_k(B) \). For quarterly data the four relevant roots, \( \theta_k \), are 1, \(-1\), \( i \), and \(-i \), which after solving for the \( \lambda \)'s becomes

\[
C(B) = \Psi_1[1 + B + B^2 + B^3] + \Psi_2[1 - B + B^2 - B^3] + (\Psi_3 + \Psi_4B)[1 - B^2] + C^{**}(B)(1 - B^4), \tag{4.2}
\]

where \( \Psi_1 = C(1)/4, \Psi_2 = C(-1)/4, \Psi_3 = \text{Re}[C(i)]/2, \) and \( \Psi_4 = \text{Im}[C(i)]/2. \) Multiplying (4.1) by a vector \( \alpha' \) gives

\[
(1 - B^4)\alpha'x_t = \alpha'C(B)e_t.
\]

Suppose for some \( \alpha = \alpha_1, \alpha_1'C(1) = 0 = \alpha_i'\Psi_1 \), then there is a factor of \((1 - B)\)
in all terms, which will cancel out giving

\[
(1 + B + B^2 + B^3) \alpha_i' x_t = \alpha_1' \left\{ \Psi_2 \left[ (1 + B^2) \right] + (\Psi_3 + \Psi_4 B) [1 + B] \\
+ C^* (B) [1 + B + B^2 + B^3] \right\} \epsilon_t,
\]

so that \( \alpha_i' x_t \) will have unit roots at the seasonal frequencies but not at zero frequency. Thus \( x \) is cointegrated at zero frequency with cointegrating vector \( \alpha_1 \), if \( \alpha_1' C(1) = 0 \). Denote these as

\[
x_t \sim CI_0 \quad \text{with cointegrating vector } \alpha_1.
\]

Notice that the vector \( y_{1t} = S(B) x_t \) is \( I_0(1) \) since \( (1 - B) y_{1t} = C(B) \epsilon_t \), while \( \alpha_1' y_{1t} \) is stationary whenever \( \alpha_1' C(1) = 0 \) so that \( y_{1t} \) is cointegrated in exactly the sense described in Engle and Granger (1987). Since \( y_{1t} \) is essentially seasonally adjusted \( x_t \) it follows that one strategy for estimation and testing for cointegration at zero frequency in seasonal series is to first seasonally adjust the series.

Similarly, letting \( y_{2t} = -(1 - B)(1 + B^2) x_t, (1 + B) y_{2t} = -C(B) \epsilon_t \) so that \( y_{2t} \) has a unit root at \( -1 \). If \( \alpha_2' C(-1) = 0 \), then \( \alpha_2' \Psi_2 = 0 \) and \( \alpha_2' y_{2t} \) will not have a unit root at \( -1 \). We say then that \( x_t \) is cointegrated at frequency \( \omega = \frac{1}{2} \), which is denoted

\[
x_t \sim CI_{1/2} \quad \text{with cointegrating vector } \alpha_2.
\]

Finally denote \( y_{3t} = -(1 - B^2) x_t \), which satisfies \( (1 + B^2) y_{3t} = -C(B) \epsilon_t \) and therefore includes unit roots at frequency \( \frac{1}{4} \). If \( \alpha_3' C(i) = 0 \) which implies that \( \alpha_3' \Psi_3 = \alpha_3' \Psi_4 = 0 \), then \( \alpha_3' y_{3t} \) will not have a unit root at \( \frac{1}{4} \), implying that

\[
x_t \sim CI_{1/4} \quad \text{with cointegrating vector } \alpha_3.
\]

Cointegration at frequency \( \frac{1}{4} \) can also occur under weaker conditions. Consider the bivariate system:

\[
(1 + B^2) x_t = \begin{bmatrix} 1 & 0 \\ B & 1 + B^2 \end{bmatrix} \epsilon_t,
\]

in which both series are \( I_{1/4}(1) \) and there is no fixed cointegrating vector. However, the polynomial cointegrating vector (PCIV), as introduced by Yoo (1987), of \( (-B, 1) \) will generate a stationary series. It is not surprising with seasonal unit roots, that the timing could make a difference. We now show that the need for PCIV is a result purely of the fact that one vector is sought to
eliminate two roots \((\pm i)\) and that one lag in the cointegrating polynomial is sufficient.

Expanding the PCIV \(a(B)\) about the two roots \((\pm i)\) using (3.2) gives

\[
a(B) = \text{Re}[a(i)] + B\text{Im}[a(i)] + a**((1 + B^2)) = (a_3 + a_4B) + a**(B)(1 + B^2),
\]

so that the condition that \(a'(B)C(B)\) have a common factor of \((1 + B^2)\) depends only on \(a_3\) and \(a_4\). The general statement of cointegration at frequency \(\frac{1}{4}\) then becomes

\[
x_t \sim CI_{1/4} \quad \text{with polynomial cointegrating vector } a_3 + a_4B,
\]

if and only if \((\alpha_3 + \alpha_4i)(\Psi_3 - \Psi_4i) = 0\),

which is equivalent to \(a(i)'C(i) = 0\).

There is no guarantee that \(x_t\) will have any type of cointegration or that these cointegrating vectors will be the same. It is however possible that \(a_1 = a_2 = a_3, a_4 = 0\), and therefore one cointegrating vector could reduce the integration of the \(x\) series at all frequencies. Similarly if \(a_2 = a_3, a_4 = 0\), one cointegrating vector will eliminate the seasonal unit roots. This might be expected if the seasonality in two series is due to the same source.

A characterization of the cointegrating possibilities has now been given in terms of the moving-average representation. More useful are the autoregressive representations and in particular, the error-correction representation. Therefore, if \(C(B)\) is a rational matrix in \(B\), it can be written [using the Smith–McMillan decomposition [Kailath (1980)], as adapted by Yoo (1987), and named the Smith–McMillan–Yoo decomposition by Engle (1987)] as follows:

\[
C(B) = U(B)^{-1}M(B)V(B)^{-1},
\]

where \(M(B)\) is a diagonal matrix whose determinant has roots only on the unit circle, and the roots of the determinants of \(U^{-1}(B)\) and \(V(B)^{-1}\) lie outside the unit circle. This diagonal could contain various combinations of the unit roots. However, assuming that the cointegrating rank at each frequency is \(r\), the matrix can be written without loss of generality as

\[
M(B) = \begin{bmatrix}
I_{N-r} & 0 \\
0 & \Delta_4I_r
\end{bmatrix},
\]

where \(I_k\) is a \(k \times k\) unit matrix. The following derivation of the error-correction representation is easily adapted for other forms of \(M(B)\).
Substituting (4.3) into (4.1) and multiplying by $U(B)$ gives

$$
\Delta_4 U(B) x_t = M(B)V(B)^{-1} \varepsilon_t. \tag{4.5}
$$

The first $N - r$ equations have a $\Delta_4$ on the left side only while the final $r$ equations have $\Delta_4$ on both sides which therefore cancel. Thus (4.5) can be written as

$$
\overline{M}(B) U(B) x_t = V(B)^{-1} \varepsilon_t, \tag{4.6}
$$

with

$$
\overline{M}(B) = \begin{bmatrix} \Delta_4 I_{N-r} & 0 \\ 0 & I_r \end{bmatrix}. \tag{4.7}
$$

Finally, the autoregressive representation is obtained by multiplying by $V(B)$ to obtain

$$
A(B) x_t = \varepsilon_t, \tag{4.8}
$$

where

$$
A(B) = V(B) \overline{M}(B) U(B). \tag{4.9}
$$

Notice that at the seasonal and zero-frequency roots, $\det[A(\theta)] = 0$ since $A(B)$ has rank $r$ at those frequencies. Now, partition $U(B)$ and $V(B)$ as

$$
U(B) = \begin{bmatrix} U_1(B) \\ \alpha(B) \end{bmatrix}, \quad V(B) = \begin{bmatrix} V_1(B), \gamma(B) \end{bmatrix},
$$

where $\alpha(B)$ and $\gamma(B)$ are $N \times r$ matrices and $U_1(B)$ and $V_1(B)$ are $N \times (N - r)$ matrices. Expanding the autoregressive matrix using (3.3) gives

$$
A(B) = \Pi_1 B \left[ 1 + B + B^2 + B^3 \right] - \Pi_2 B \left[ (1 - B)(1 + B^2) \right]
$$

$$
+ \left( \Pi_4 - B \Pi_3 \right) B [1 + B^2] + A^*(B) [1 - B^4],
$$

with $\Pi_1 = -\gamma(1)\alpha'(1)/4 \equiv -\gamma_1 \alpha_1'$, $\Pi_2 = -\gamma(-1)\alpha(-1)'/4 \equiv -\gamma_2 \alpha_2'$, $\Pi_3 = \text{Re}[\gamma(i)\alpha(i)']/2$, and $\Pi_4 = \text{Im}[\gamma(i)\alpha(i)']/2$. Letting $\alpha_1 = \alpha(1)/4$, $\alpha_2 = \alpha(-1)/4$, $\alpha_3 = \text{Re}[\alpha(i)]/2$, and $\alpha_4 = \text{Im}[\alpha(i)]$ while $\gamma_1 = \gamma(1)$, $\gamma_2 = \gamma(-1)$, $\gamma_3 = \text{Re}[\gamma(i)]$, and $\gamma_4 = \text{Im}[\gamma(i)]$, the general error-correction model can be
written

\[
A^*(B) \Delta_4 x_t = \gamma_1 \alpha_1' y_{1t-1} + \gamma_2 \alpha_2' y_{2t-1} - (\gamma_3 \alpha_3' - \gamma_4 \alpha_4') y_{3t-2} + (\gamma_4 \alpha_3' + \gamma_3 \alpha_4') y_{3t-1} + \epsilon_t,
\]

(4.10)

where \( A^*(0) = C(0) = I_N \) in the standard case. This expression is an error-correction representation where both \( \alpha \), the cointegrating vector, and \( \gamma \), the coefficients of the error-correction term, may be different at different frequencies and, in one case, even at different lags. This can be written in a more transparent form by allowing more than two lags in the error-correction term. Add \( \Delta_4(\gamma_3 \alpha_4' + \gamma_4 \alpha_3' + \gamma_4 \alpha_4' B)x_{t-1} \) to both sides and rearrange terms to get

\[
\tilde{A}^*(B) \Delta_4 x_t = \gamma_1 \alpha_1' y_{1t-1} + \gamma_3 \alpha_2' y_{2t-1} - (\gamma_3 + \gamma_4 B)(\alpha_3' + \alpha_4' B) y_{3t-2} + \epsilon_t,
\]

(4.11)

where \( \tilde{A}^*(B) \) is a slightly different autoregressive matrix from \( A^*(B) \). The error-correction term at the annual seasonal enters potentially with two lags and is potentially a polynomial cointegrating vector. When \( \alpha_3 = 0 \) or \( \alpha_4 = 0 \) or both, the model simplifies so that, respectively, cointegration is contemporaneous, the error correction needs only one lag, or both.

Notice that all the terms in (4.11) are stationary. Estimation of the system is easily accomplished if the \( \alpha \)'s are known a priori. If they must be estimated, it appears that a generalization of the two-step estimation procedure proposed by Engle and Granger (1987) is available. Namely, estimate the \( \alpha \)'s using prefiltered variables \( y_1, y_2, \) and \( y_3 \), respectively, and then estimate the full model using the estimates of the \( \alpha \)'s. In the PCIV case this regression would include a single lag. It is conjectured that the least-squares estimates of the remaining parameters would have the same limiting distribution as the estimator knowing the true \( \alpha \)'s just as in the Engle-Granger two-step estimator. The analysis by Stock (1987) suggests that although the inference on the \( \alpha \)'s can be tricky due to their nonstandard limiting distributions, inference on the estimates of \( A^*(B) \) and the \( \gamma \)'s can be conducted in the standard way.

The following generalizations of the above analysis are discussed formally in Yoo (1987). First, if \( r > 1 \) but all other assumptions remain as before, the error-correction representation (4.11) remains the same but the \( \alpha \)'s and \( \gamma \)'s now become \( N \times r \) matrices. Second, if the cointegrating rank at the long-run frequency is \( r_0 \), which is different from the cointegrating rank at the seasonal frequency, \( r_s \), (4.11) is again legitimate with the sizes of the matrices on the right-hand side appropriately redefined. Thirdly, if the cointegrating vectors
\( \alpha_1, \alpha_2, \) and \( \alpha_3 \) coincide, equalling say, \( \alpha, \) and \( \alpha_4 = 0, \) a simpler error-cointegrating model occurs:

\[
A^*(B)\Delta x_t = \gamma(B)\alpha'x_{t-1} + \varepsilon_t,
\]  

where the degree of \( \gamma(B) \) is at most 3, as can be seen either from (4.10) or from an expansion of \( \gamma(B) \) using (3.2). For four roots there are potentially four coefficients and three lags.

Finally, some of the cointegrating vectors may coincide but some do not. A particularly interesting case is where a single linear combination eliminates all seasonal unit roots. Thus suppose \( \alpha_2 = \alpha_3 = \alpha \) and \( \alpha_4 = 0. \) Then (4.10) becomes

\[
A^*(B)\Delta x_t = \gamma_1\alpha'\Delta x_{t-1} + \gamma_2(B)\alpha'\Delta x_{t-1} + \varepsilon_t,
\]

where \( \gamma_2(B) \) has potentially two lags. Thus zero-frequency cointegration occurs between the elements of seasonally adjusted \( x, \) while seasonal cointegration occurs between the elements of differenced \( x. \) This is the case examined by Engle, Granger, and Hallman (1989) for electricity demand. There monthly electricity sales were modeled as cointegrated with economic variables such as customers and income at zero frequency and possibly at seasonal frequencies with the weather. The first relation is used in long-run forecasting, while the second is mixed with the short-run dynamics for short-run forecasting.

Although an efficiency gain in the estimates of the cointegrating vectors is naturally expected by checking and imposing the restrictions between the cointegrating vectors, there should be no efficiency gain in the estimates of the ‘short-run parameters’, namely \( A^*(B) \) and \( \gamma \)'s, given the superconsistency of the estimates of the cointegrating vectors. Hence, the representation (4.11) is considered relatively general and the important step of model-building procedure is then to identify the cointegratedness at the different frequencies. This question is considered in the next section.

5. Testing for cointegration: An application

In this section it is assumed that there are two series of interest, \( x_{1t} \) and \( x_{2t}, \) both integrated at some of the zero and seasonal frequencies, and the question to be studied is whether or not the series are cointegrated at some frequency. Of course, if the two series do not have unit roots at corresponding frequencies, the possibility of cointegration does not exist. The tests discussed in section 3 can be used to detect which unit roots are present.

Suppose for the moment that both series contain unit roots at the zero frequency and at least some of the seasonal frequencies. If one is interested in
the possibility of cointegration at the zero frequency, a strategy could be to form the static O.L.S. regression

\[ x_{1t} = A x_{2t} + \text{residual}, \]

and then test if the residual has a unit root at zero frequency, which is the procedure in Engle and Granger (1987). However, the presence of seasonal unit roots means that \( A \) may not be consistently estimated, in sharp contrast to the case when there are no seasonal roots when \( A \) is estimated superefficiently. This lack of consistency is proved in Engle, Granger, and Hallman (1989). If, in fact, \( x_{1t} \) and \( x_{2t} \) are cointegrated at both the zero and the seasonal frequencies, with cointegrating vectors \( \alpha_1 \) and \( \alpha_s \) and with \( \alpha_1 \neq \alpha_s \), it is unclear what value of \( A \) would be chosen by the static regression. Presumably, if \( \alpha_1 = \alpha_s \), then \( \hat{A} \) will be an estimate of this common value. These results suggest that the standard procedure for testing for cointegration is inappropriate.

An alternative strategy would be to filter out unit-root components other than the one of interest and to test for cointegration with the filtered series. For example, to remove seasonal roots, one could form

\[ \tilde{x}_{1t} = S(B)x_{1t}, \quad \tilde{x}_{2t} = S(B)x_{2t}, \]

where \( S(B) = (1 - B^s)/(1 - B) \), and then perform a standard cointegration test, such as those discussed in Engle and Granger (1987), on \( \tilde{x}_{1t} \) and \( \tilde{x}_{2t} \). If some seasonal unit roots were thought to be present in \( x_{1t} \) and \( x_{2t} \), this procedure could be done without testing for which roots were present, but the filtered series could have spectra with zeros at some seasonal frequencies, and this may introduce problems with the tests. Alternatively, the tests of section 3 could be used, appropriate filters applied just to remove the seasonal roots indicated by these tests, and then the standard cointegration tests applied. For zero-frequency cointegration, this procedure is probably appropriate, although the implications of the pretesting for seasonal roots has not yet been investigated.

To test for seasonal cointegration the corresponding procedure would be to difference the series to remove a zero-frequency unit root, if required, then run a regression of the form

\[ \Delta x_{1t} = \sum_{j=0}^{s-2} \alpha_j \Delta x_{2t-j} + \text{residual}, \]

and test if the residual has any seasonal unit roots. The tests developed in section 3 could be applied, but will not have the same distribution as they involve estimates of the \( \alpha_j \). The correct test has yet to be developed.

A situation where the tests of section 3 can be applied directly is where \( \alpha_1 = \alpha_s \) and some theory suggests a value for this \( \alpha \), so that no estimation is
Fig. 1. Income and consumption in the UK.
required. One merely forms $x_{1t} - \alpha x_{2t}$ and tests for unit roots at the zero and seasonal frequencies.

An example comes from the permanent income hypothesis where the log of income and the log of consumption may be thought to be cointegrated with $\alpha = 1$. Thus $c - y$ should have no unit roots using a simplistic form of this theory, as discussed by Davidson, Hendry, Srba, and Yeo (1978), for instance.

To illustrate the tests, quarterly United Kingdom data for the period 1955.1 to 1984.4 were used with $y = \log$ of personal disposable income and $c = \log$ of consumption expenditures on nondurables. The data are shown in fig. 1.

From the figure, it is seen that both series may have a random-walk character implying that we would expect to find a unit root at the zero frequency. However, the two series seem to drift apart whereby cointegration at the zero frequency with cointegrating vector $(1, -1)$ is less likely. For the seasonal pattern, it is clear that $c$ contains a much stronger and less changing seasonal pattern than $y$, although even the seasonal consumption pattern changes over the sample period. Based on these preliminary findings, one may or may not find seasonal unit roots in $c$ and $y$ or both, but cointegration at the seasonal frequencies cannot be expected.

The tests are based on the auxiliary regression (3.6) where $\phi(B)$ is a polynomial in $B$. The deterministic term is a zero, an intercept ($I$), an intercept and seasonal dummies ($I$, $SD$), an intercept and a trend ($I$, $Tr$), or an intercept, seasonal dummies, and a trend ($I$, $SD$, $Tr$).

In the augmented regressions nonsignificant lags were removed, and for $c$ and $y$ this implied a lag polynomial of the form $1 - \phi_1 B - \phi_4 B^4 - \phi_5 B^5$, where $\phi_1$ was around 0.85, $\phi_4$ around -0.32, and $\phi_5$ around 0.25. For $c - y$ the lag polynomial was approximately $1 - 0.29 B - 0.22 B^2 + 0.21 B^4$. The 't' statistics from these augmented regressions are shown in table 2.

The results indicate strongly a unit root at the zero frequency in both $c$, $y$, and $c - y$ implying that there is no cointegration between $c$ and $y$ at the long-run frequency, at least not for the cointegrating vector $(1, -1)$.

Similarly, the hypothesis that $c$, $y$, and $c - y$ are $I_{1/2}(1)$ cannot be rejected implying that $c$ and $y$ are not cointegrated at the biannual cycle either.

The results also indicate that the log of consumption expenditures on nondurables are $I_{1/4}(1)$ as neither the ‘F’ test nor the two ‘t’ tests can reject the hypothesis that both $\pi_4$ and $\pi_3$ are zero. Such hypotheses are, however, firmly rejected for the log of personal disposable income and conditional on these results, $c$ and $y$ cannot possibly be cointegrated at this frequency or at the frequency corresponding to the complex conjugate root, irrespective of the forms of the cointegrating vectors. In fact, conditional on $\pi_4$ being zero, the ‘t’ test on $\pi_3$ cannot reject a unit root in $c - y$ at the annual frequency in any of the auxiliary regressions. The assumption that $\pi_4 = 0$ is not rejected when seasonal dummies are absent and the joint ‘F’ test cannot reject in these cases either. When the auxiliary regression contains deterministic seasonals, both
Table 2
Tests for seasonal unit roots in the log of UK consumption expenditure on nondurables $c$, in the log of personal disposable income $y$, and in the difference $c - y$, 1955.1–1984.4.

<table>
<thead>
<tr>
<th>VAR</th>
<th>Auxiliary regression</th>
<th>$t'$: $\pi_1$ (zero frequency)</th>
<th>$t'$: $\pi_2$ (biannual)</th>
<th>$t'$: $\pi_3$ (annual)</th>
<th>$t'$: $\pi_4$</th>
<th>$F'$: $\pi_3 \cap \pi_4$</th>
</tr>
</thead>
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<tr>
<td>$c$</td>
<td>$I$</td>
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<td>0.22</td>
<td>-0.84</td>
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<td>0.38</td>
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<td></td>
<td>$I, SD, Tr$</td>
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<td>-2.16</td>
<td>-1.53</td>
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<tr>
<td>$y$</td>
<td>$I$</td>
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<td>7.87</td>
</tr>
</tbody>
</table>

a The auxiliary regressions were augmented by significant lagged values of the fourth difference of the regressand.

b Significant at the 5% level.

$\pi_4 = 0$ and $\pi_3 \cap \pi_4 = 0$ are rejected leading to a theoretical conflict which can of course happen with finite samples.

6. Conclusion

The theory of integration and cointegration of time series is extended to cover series with unit roots at frequencies different from the long-run frequency. In particular, seasonal series are studied with a focus upon the quarterly periodicity. It is argued that the existence of unit roots at the seasonal frequencies has similar implications for the persistence of shocks as a unit root at the long-run frequency. However, a seasonal pattern generated by a model characterized solely by unit roots seems unlikely as the seasonal pattern becomes too volatile, allowing 'summer to become winter'.

A proposition on the representation of rational polynomials allows reformulation of an autoregression isolating the key unit-root parameters. Based on least-squares fits of univariate autoregressions on transformed variables, similar to the well-known augmented Dickey–Fuller regression, tests for the existence of seasonal as well as zero-frequency unit roots in quarterly data are presented and tables of the critical values provided.

By extending the definition of cointegration to occur at separate frequencies, the error-correction representation is developed by use of the Smith–
McMillan lemma and the proposition on rational lag polynomials. The error-correction representation is shown to be a direct generalization of the well-known form, but on properly transformed variables.

The theory is applied to the UK consumption function and it is shown that the unit-elasticity error-correction model is not valid at any frequency as long as we confine ourselves to only the consumption and income data.

References


Yoo, S., 1987, Co-integrated time series: Structure, forecasting and testing, Ph.D. dissertation (University of California, San Diego, CA).